Software Model-Checking and Cyclic Proof Search

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This talk

Relationship between **software model checking** and **cyclic proof search**

[Ball+ 2001] [Birgmeier+ 2014]
[Cimatti&Griggio 2012] [Cimatti+ 2014]
[Henzinger+ 2004, 2002] [Hoder&Bjørner 2012]
[Komuravelli+ 2014, 2013] [McMillan 2006] …

[Brotherston and Simpson 2011]
[Sprenger and Dam 2003] …

Known **Model-checking problem** $\leftrightarrow$ **Validity/(un)satisfiability problem**

New

Model-checking algorithms

= proof search heuristics

(**Internal states of algorithms** = **partially constructed proofs**)

Our aim from the viewpoint of software model-checking

Providing a **unified account for model-checking algorithms** in terms of **logic**

- To **understand** behaviours of many algorithms using a **single common structure**

  partially constructed proofs

- To **compare** different algorithms
  - Property-directed reachability ≈ Efficient game solving algorithm

    [Bradley 2011] [Een+ 2011] [Cimatti&Griggio 2012] [Farzan&Kincaid 2017]

- To **develop** new algorithms
  - Refutationally complete variant of PDR

Our aim from the viewpoint of cyclic proof search

Importing ideas and techniques of software model-checking to cyclic proof search

- **Finding an appropriate cut formula is crucial** for cyclic proof search
  - **Cut-elimination fails** for cyclic proof systems [Kimura+ 2020] [Masuoka&Tatsuta 2021]

- Software model-checking community has developed
  highly-efficient algorithms to find an appropriate cut formula
  - Existing proof search strategies for cyclic proof system ≈ bounded model-checking + covering
    E.g. [Tellez and Brotherston 2020]

Outline

- **Background**
    - **Software model-checking**
    - Proof systems for inductive definitions
- Key observation
- Software model-checking as cyclic proof search

Software model-checking

Algorithmic analysis of programs to prove properties of their executions

[Jhara&Majumdar 2009]

Let us focus on **safety verification of a while program**

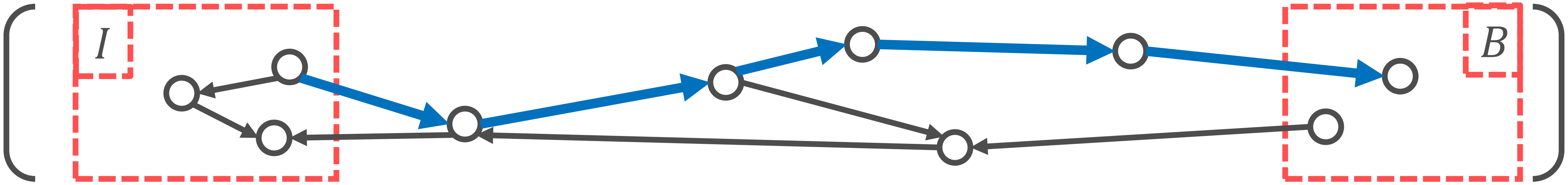
Input	Set of **states**	D
	Initial states	$I \subseteq D$
	Bad states	$B \subseteq D$
	Transition relation	$T \subseteq D \times D$

Usually **infinite**, e.g. $D = \mathbb{Z}^n$

Output Whether B is **unreachable** from I via T

- $\neg \exists s_0 s_1 \ldots s_n \in S. I(s_0) \wedge T(s_0, s_1) \wedge \cdots \wedge T(s_{n-1}, s_n) \wedge B(s_n)$

$\neg$



Inductive invariant

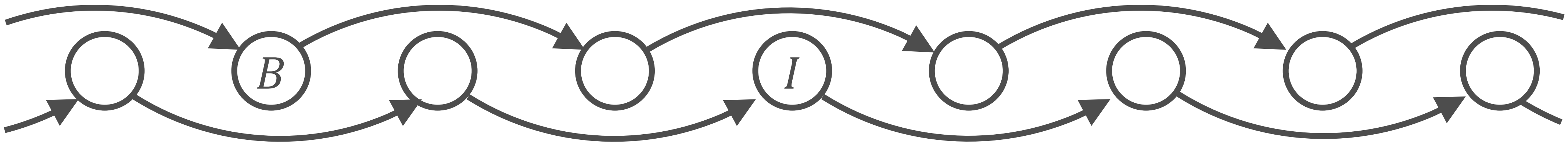
Set of states	D
Initial states	$I \subseteq D$
Bad states	$B \subseteq D$
Transition relation	$T \subseteq D \times D$

A witness of the safety of a given system

Def A subset $P \subseteq D$ is an **inductive invariant** if

- all initial states are P $\qquad I(x) \Longrightarrow P(x)$
- P contains no bad state $\qquad P(x) \Longrightarrow \neg B(x)$
- P is closed under the transition relation $\qquad P(x) \wedge T(x, y) \Longrightarrow P(y)$

Example $\quad D = \mathbb{Z},\ I = \{0\},\ B = \{-3\},\ T = \{\ (n, n+2) \mid n \in \mathbb{Z}\ \}$



- $P_1 = \{\ 2n \mid n \in \mathbb{Z}, n \geq 0\ \}$ is an inductive invariant
- $P_2 = \{\ n \in \mathbb{Z} \mid n \geq 0\ \}$ is an inductive invariant

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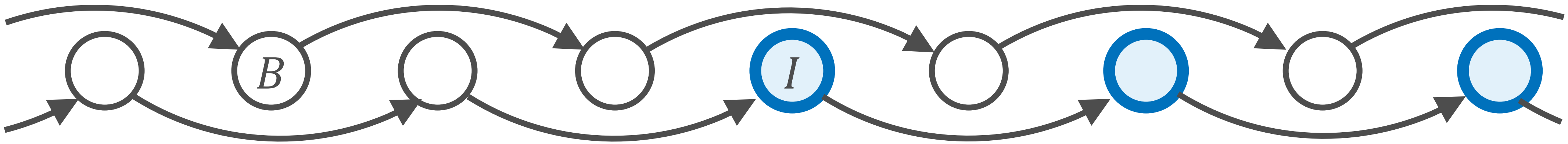

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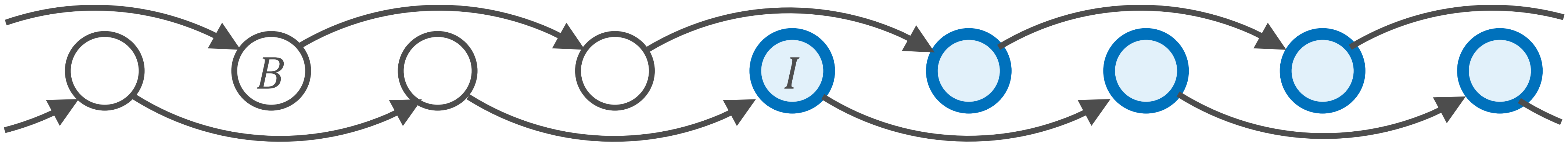

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Prop If an inductive invariant $P \subseteq D$ exists, the system never reaches a bad state

Model-checkers search for inductive invariants in a variety of cleaver ways

- It is **relatively easy to check** if a given $P \subseteq D$ is indeed an inductive invariant

Outline

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- Key observation
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A Logical Formalisation

The **set of reachable states** is the **least solution μR** for P in

$$P(x) \quad \stackrel{\mu}{\Longleftrightarrow} \quad I(x) \vee \left(\exists y. P(y) \wedge T(y, x)\right)$$

- Defining a property as the least solution of an equation = **inductive definition**

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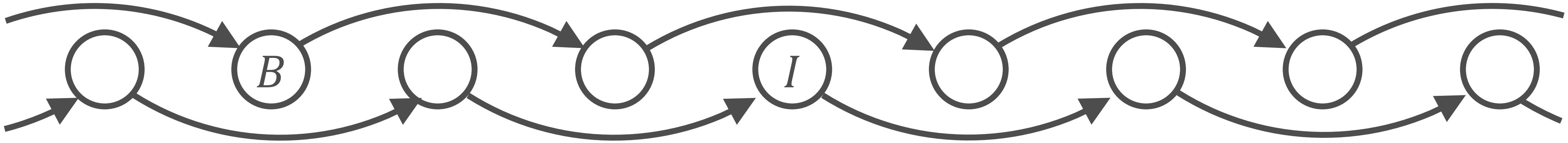

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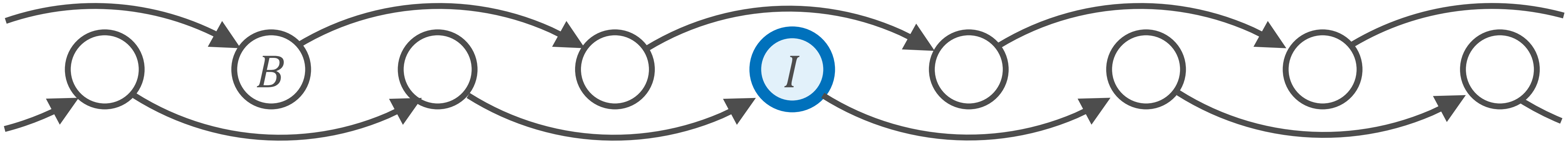

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The **set of reachable states** is the **least solution μR** for P in

$$P(x) \quad \overset{\mu}{\Longleftrightarrow} \quad I(x) \lor \left(\exists y. P(y) \land T(y, x)\right)$$

- Defining a property as the least solution of an equation = **inductive definition**

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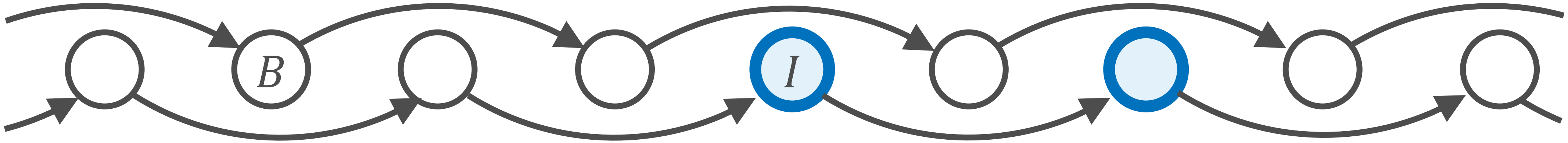

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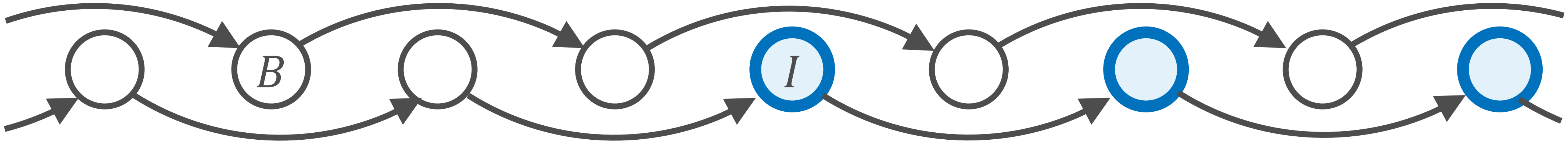

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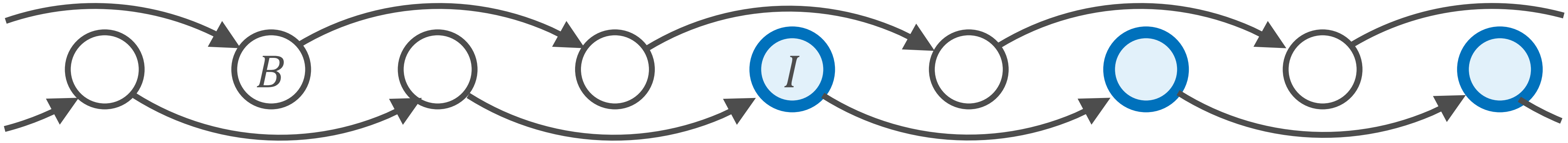

$$\mu R = \{2n \mid n \in \mathbb{Z}, n \geq 0\}$$

A Logical Formalisation

The **set of reachable states** is the **least solution μR** for P in

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- Defining a property as the least solution of an equation = **inductive definition**

<u>**Prop**</u> The system never reaches a bad state if and only if **$\mu R(x) \vdash \neg B(x)$ is valid**

- Simply because μR is the set of reachable states

Proof systems for inductive definitions are usable to prove $\mu R(x) \vdash \neg B(x)$

Classical proof rule for inductive definitions

Due to Martin-Löf (1972)

$$\boxed{\mu R(x) \overset{\mu}{\iff} I(x) \lor (\exists y. \mu R(y) \land T(y,x))}$$

$$\frac{I(x) \lor (\exists y. \varphi(y) \land T(y,x)) \vdash \varphi(x) \qquad \varphi(x) \vdash \neg B(x)}{\mu R(x) \vdash \neg B(x)}$$

The premises require that **$\varphi(x)$ is an inductive invariant**

- Initial states satisfy φ — $I(x) \vdash \varphi(x)$
- φ is closed under the transition — $\exists y. \varphi(y) \land T(y,x) \vdash \varphi(x)$
- φ has no bad state — $\varphi(x) \vdash \neg B(x)$

This rule cannot be used to describe processes searching for inductive invariants

- This rule is **applicable only after an inductive invariant φ is found**

Cyclic proof system

[Brotherston&Simpson 2011]
[Sprenger&Dam 2003] …

A proof system in which **proofs may have cycles**

- **Cycle ≈ use of induction hypothesis**

$$
\dfrac{
  \dfrac{
    \dfrac{\vdots}{I(x) \vdash \varphi(x)}
    \qquad
    \dfrac{
      \dfrac{
        \boxed{\mu R(y) \vdash \varphi(y)}
        \qquad
        \dfrac{\vdots}{\varphi(y) \vdash T(y,x) \Rightarrow \varphi(x)}
      }{\mu R(y) \vdash T(y,x) \Rightarrow \varphi(x)}
    }{\exists y.\mu R(y) \wedge T(y,x) \vdash \varphi(x)}
  }{
    \dfrac{I(x) \vee (\exists y.\mu R(y) \wedge T(y,x)) \vdash \varphi(x)}{\mu R(x) \vdash \varphi(x)}
  }
  \qquad
  \dfrac{\vdots}{\varphi(x) \vdash \neg B(x)}
}{\mu R(x) \vdash \neg B(x)}
$$

A rule for inductive definition just expands the definition

- **Applicable without knowing an inductive invariant**

$$\dfrac{I(x) \vee (\exists y.\mu R(y) \wedge T(y,x)) \vdash \varphi(x)}{\mu R(x) \vdash \varphi(x)}$$

$$\boxed{\mu R(x) \overset{\mu}{\Longleftrightarrow} I(x) \vee (\exists y.\mu R(y) \wedge T(y,x))}$$

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- **Key observation**
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Key observation

To establish a precise connection between model-checking and proof search,

- **"all reachable states are not bad"** is inappropriate,

$$\mu R(x) \vdash \neg B(x) \qquad\qquad \mu R(x) \overset{\mu}{\iff} I(x) \vee \big(\exists y. \mu R(y) \wedge T(y, x)\big)$$

  - A state x is **reachable** if $\exists y_0 y_1 \dots y_{n-1}. I(y_0) \wedge T(y_0, y_1) \wedge \cdots \wedge T(y_{n-1}, x)$
    (cf. **strongest post-condition**, **backward reachability checking**)

- but the **dual** formalisation **"all initial states are safe"** should be used

$$I(x) \vdash \nu S(x) \qquad\qquad \nu S(x) \overset{\nu}{\iff} \neg B(x) \wedge \big(\forall y. T(x, y) \Rightarrow S(y)\big)$$

greatest solution

  - A state x is **safe** if $\neg\exists y_1 \dots y_n. T(x, y_1) \wedge \cdots \wedge T(y_{n-1}, y_n) \wedge B(y_n)$
    (cf. **weakest pre-condition**, **forward reachability checking**)

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Goal-oriented proof search

A bottom-up proof-search

- An intermediate state is a **proof with unproved leaves**

1. Start from the tree consisting only of the **goal sequent**

$$I(x) \overset{?}{\vdash} \nu S(x)$$

2. Choose an unproved leaf and **select an appropriate proof rule** for it

$$\frac{I(x) \overset{?}{\vdash} \neg B(x) \wedge \left(\forall y. T(x,y) \Rightarrow \nu S(y)\right)}{I(x) \vdash \nu S(x)}$$

3. Iterate this process until there are no unproved leaves

Symbolic execution

<u>**Heuristic 1**</u> Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

$$\frac{I(x) \overset{?}{\vdash} \neg B(x) \wedge \big(\forall y. T(x,y) \Rightarrow \nu S(y)\big)}{I(x) \vdash \nu S(x)}$$

Symbolic execution

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$$\dfrac{\dfrac{I(x) \overset{?}{\vdash} \neg B(x) \qquad\qquad I(x) \overset{?}{\vdash} \forall y. T(x,y) \Rightarrow \nu S(y)}{I(x) \vdash \neg B(x) \wedge \big(\forall y. T(x,y) \Rightarrow \nu S(y)\big)}}{I(x) \vdash \nu S(x)}$$

Symbolic execution

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

An SMT solver can automatically (dis)prove

$$\dfrac{\dfrac{I(x) \overset{?}{\vdash} \neg B(x) \qquad\qquad I(x) \overset{?}{\vdash} \forall y. T(x,y) \Rightarrow \nu S(y)}{I(x) \vdash \neg B(x) \land \big(\forall y. T(x,y) \Rightarrow \nu S(y)\big)}}{I(x) \vdash \nu S(x)}$$

Symbolic execution

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

$$\dfrac{\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \overset{\textbf{?}}{I(x) \vdash \forall y. T(x,y) \Rightarrow \nu S(y)}}{I(x) \vdash \neg B(x) \wedge \big(\forall y. T(x,y) \Rightarrow \nu S(y)\big)}}{I(x) \vdash \nu S(x)}$$

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Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

$$\dfrac{\dfrac{\begin{array}{c}\text{SMT}\\ I(x) \vdash \neg B(x)\end{array} \qquad \dfrac{I(x) \overset{?}{\vdash} T(x,y) \Rightarrow \nu S(y)}{I(x) \vdash \forall y.T(x,y) \Rightarrow \nu S(y)}}{I(x) \vdash \neg B(x) \wedge \left(\forall y.T(x,y) \Rightarrow \nu S(y)\right)}}{I(x) \vdash \nu S(x)}$$

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$$\dfrac{\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \dfrac{\dfrac{I(x), T(x,y) \overset{?}{\vdash} \nu S(y)}{I(x) \vdash T(x,y) \Rightarrow \nu S(y)}}{I(x) \vdash \forall y. T(x,y) \Rightarrow \nu S(y)}}{I(x) \vdash \neg B(x) \wedge \big(\forall y. T(x,y) \Rightarrow \nu S(y)\big)}}{I(x) \vdash \nu S(x)}$$

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Symbolic execution

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One-step transition

$$
\dfrac{
\begin{array}{c}\text{SMT}\\ I(x) \vdash \neg B(x)\end{array}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\boxed{\exists x. I(x) \wedge T(x,y)}^{\,?} \vdash \nu S(y)}{I(x), T(x,y) \vdash \nu S(y)}}{I(x) \vdash T(x,y) \Rightarrow \nu S(y)}}{I(x) \vdash \forall y. T(x,y) \Rightarrow \nu S(y)}}
{\dfrac{I(x) \vdash \neg B(x) \wedge \left(\forall y. T(x,y) \Rightarrow \nu S(y)\right)}{\boxed{I(x)} \vdash \nu S(x)}}
$$

Symbolic execution

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Next states of I are safe

Initial states are not bad

One-step transition

?

$$\dfrac{\dfrac{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \dfrac{\dfrac{\dfrac{\boxed{\exists x. I(x) \wedge T(x,y)} \vdash \nu S(y)}{I(x), T(x,y) \vdash \nu S(y)}}{I(x) \vdash T(x,y) \Rightarrow \nu S(y)}}{I(x) \vdash \forall y. T(x,y) \Rightarrow \nu S(y)}}{\dfrac{I(x) \vdash \neg B(x) \wedge (\forall y. T(x,y) \Rightarrow \nu S(y))}{\boxed{I(x)} \vdash \nu S(x)}}$$

Symbolic execution

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Next states of φ are safe

Current states are not bad

One-step transition

$$\dfrac{\dfrac{\mathrm{SMT}}{\varphi(x) \vdash \neg B(x)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{?}{\boxed{\exists x.\varphi(x) \land T(x,y)}} \vdash \nu S(y)}{\varphi(x), T(x,y) \vdash \nu S(y)}}{\varphi(x) \vdash T(x,y) \Rightarrow \nu S(y)}}{\varphi(x) \vdash \forall y.T(x,y) \Rightarrow \nu S(y)}}{\dfrac{\varphi(x) \vdash \neg B(x) \land (\forall y.T(x,y) \Rightarrow \nu S(y))}{\boxed{\varphi(x)} \vdash \nu S(x)}}$$

Generalise to arbitrary set φ

Symbolic execution

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

$$\dfrac{\dfrac{\text{SMT}}{\varphi(x) \vdash \neg B(x)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\exists x.\varphi(x) \land T(x,y) \overset{?}{\vdash} \nu S(y)}{\varphi(x), T(x,y) \vdash \nu S(y)}}{\varphi(x) \vdash T(x,y) \Rightarrow \nu S(y)}}{\varphi(x) \vdash \forall y.T(x,y) \Rightarrow \nu S(y)}}{\dfrac{\varphi(x) \vdash \neg B(x) \land \left(\forall y.T(x,y) \Rightarrow \nu S(y)\right)}{\varphi(x) \vdash \nu S(x)}}$$

A derived rule: $$\frac{\varphi(x) \vdash \neg B(x) \qquad \exists x.\varphi(x) \land T(x,y) \vdash \nu S(y)}{\varphi(x) \vdash \nu S(x)}\ (\textsc{SymbolicExecution})$$

Symbolic execution

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

$$
\dfrac{
\dfrac{
\overset{\text{SMT}}{\varphi(x) \vdash \neg B(x)}
\qquad
\dfrac{
\dfrac{
\dfrac{
\exists x.\varphi(x) \wedge T(x,y) \overset{?}{\vdash} \nu S(y)
}{
\overline{\varphi(x), T(x,y) \vdash \nu S(y)}
}
}{
\varphi(x) \vdash T(x,y) \Rightarrow \nu S(y)
}
}{
\varphi(x) \vdash \forall y.T(x,y) \Rightarrow \nu S(y)
}
}{
\varphi(x) \vdash \neg B(x) \wedge \big(\forall y.T(x,y) \Rightarrow \nu S(y)\big)
}
}{
\varphi(x) \vdash \nu S(x)
}
$$

A derived rule:

$$
\dfrac{\varphi(x) \vdash \neg B(x) \qquad \exists x.\varphi(x) \wedge T(x,y) \vdash \nu S(y)}{\varphi(x) \vdash \nu S(x)}\,(\mathrm{SE})
$$

Bounded model-checking [Biere+ 1999]

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

***k*-th iteration of (SE)** rule coincides with **model-checking within *k* steps**

$$\dfrac{\overset{\text{SMT}}{\varphi_k(x) \vdash \neg B(x)} \qquad \overset{\textbf{?}}{\varphi_{k+1}(x) \vdash \nu S(x)}}{\varphi_k(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

$$\vdots$$

$$\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \vdots}{\varphi_2(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

$$\dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \varphi_2(x) \vdash \nu S(x)}{\varphi_1(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \varphi_1(x) \vdash \nu S(x)}{I(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Forward criterion [Sheeran+ 2000]

<u>Heuristic 1</u> Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Trying to make cycles after k-th iteration of (SE) rule

$$\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \varphi_{k+1}(x) \overset{?}{\vdash} \nu S(x)}{\varphi_k(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

$$\vdots$$

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \vdots}{\varphi_2(x) \vdash \nu S(x)}\ (\mathrm{SE})}{\varphi_1(x) \vdash \nu S(x)}\ (\mathrm{SE})}{I(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Forward criterion [Sheeran+ 2000]

<u>Heuristic 1</u> Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Trying to make cycles after k-th iteration of (SE) rule

if $\varphi_{k+1}(x) = \varphi_2(x)$

$$\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \varphi_{k+1}(x) \overset{?}{\vdash} \nu S(x)}{\varphi_k(x) \vdash \nu S(x)}\ (\text{SE})$$

$$\vdots$$

$$\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \vdots}{\varphi_2(x) \vdash \nu S(x)}\ (\text{SE})$$

$$\dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \varphi_2(x) \vdash \nu S(x)}{\varphi_1(x) \vdash \nu S(x)}\ (\text{SE})$$

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \varphi_1(x) \vdash \nu S(x)}{I(x) \vdash \nu S(x)}\ (\text{SE})$$

Forward criterion [Sheeran+ 2000]

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Trying to make cycles after k-th iteration of (SE) rule

if $\varphi_{k+1}(x) = \varphi_2(x)$

$$\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \varphi_{k+1}(x) \vdash \nu S(x)}{\varphi_k(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \vdots}{\varphi_2(x) \vdash \nu S(x)}\ (\mathrm{SE})}{\varphi_1(x) \vdash \nu S(x)}\ (\mathrm{SE})}{I(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Forward criterion [Sheeran+ 2000]

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Trying to make cycles after k-th iteration of (SE) rule

$$\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \varphi_{k+1}(x) \overset{?}{\vdash} \nu S(x)}{\varphi_k(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

$$\vdots$$

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \vdots}{\varphi_2(x) \vdash \nu S(x)}\ (\mathrm{SE})}{\varphi_1(x) \vdash \nu S(x)}\ (\mathrm{SE})}{I(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Induction hypothesis

Induction hypothesis

Induction hypothesis

Forward criterion [Sheeran+ 2000]

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Trying to make cycles after k-th iteration of (SE) rule

$$\dfrac{\boxed{\varphi_{k+1}(x) \overset{?}{\vdash} \bigvee_{i=1}^{k} \varphi_i(x)} \qquad ?}{\varphi_{k+1}(x) \vdash \nu S(x)}$$

$$\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \varphi_{k+1}(x) \vdash \nu S(x)}{\varphi_k(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Induction hypothesis

$$\vdots$$

$$\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \vdots}{\varphi_2(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Induction hypothesis

$$\dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \varphi_2(x) \vdash \nu S(x)}{\varphi_1(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Induction hypothesis

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \varphi_1(x) \vdash \nu S(x)}{I(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Forward criterion [Sheeran+ 2000]

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Trying to make cycles after k-th iteration of (SE) rule

$$\dfrac{\boxed{\overset{\text{SMT}}{\varphi_{k+1}(x) \vdash \bigvee_{i=1}^{k} \varphi_i(x)}} \qquad \textbf{?}}{\varphi_{k+1}(x) \vdash \nu S(x)}$$

$$\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \varphi_{k+1}(x) \vdash \nu S(x)}{\varphi_k(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Induction hypothesis

$$\vdots$$

$$\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \vdots}{\varphi_2(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Induction hypothesis

$$\dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \varphi_2(x) \vdash \nu S(x)}{\varphi_1(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Induction hypothesis

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \varphi_1(x) \vdash \nu S(x)}{I(x) \vdash \nu S(x)}\ (\mathrm{SE})$$

Forward criterion [Sheeran+ 2000]

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Trying to make cycles after k-th iteration of (SE) rule

$$
\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \dfrac{\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \dfrac{\boxed{\overset{\text{SMT}}{\varphi_{k+1}(x) \vdash \bigvee_{i=1}^{k} \varphi_i(x)}} \qquad \bigvee_{i=1}^{k} \varphi_i(x) \overset{?}{\vdash} \nu S(x)}{\varphi_{k+1}(x) \vdash \nu S(x)}\,(\textsc{Cut})}{\varphi_k(x) \vdash \nu S(x)}\,(\mathrm{SE})}{\vdots}}{\varphi_2(x) \vdash \nu S(x)}\,(\mathrm{SE})}{\varphi_1(x) \vdash \nu S(x)}\,(\mathrm{SE})}{I(x) \vdash \nu S(x)}\,(\mathrm{SE})
$$

Induction hypothesis

Induction hypothesis

Induction hypothesis

Forward criterion [Sheeran+ 2000]

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Trying to make cycles after k-th iteration of (SE) rule

$$
\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad
\dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad
\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad
\begin{array}{c}
\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad
\dfrac{\boxed{\overset{\text{SMT}}{\varphi_{k+1}(x) \vdash \bigvee_{i=1}^{k} \varphi_i(x)}} \qquad
\dfrac{\varphi_k(x) \overset{?}{\vdash} \nu S(x) \quad \cdots \quad \varphi_1(x) \overset{?}{\vdash} \nu S(x)}{\bigvee_{i=1}^{k} \varphi_i(x) \vdash \nu S(x)}}{\varphi_{k+1}(x) \vdash \nu S(x)}\,(\textsc{Cut})}{\varphi_k(x) \vdash \nu S(x)}\,(\text{SE}) \\
\vdots
\end{array}}{\varphi_2(x) \vdash \nu S(x)}\,(\text{SE})}{\varphi_1(x) \vdash \nu S(x)}\,(\text{SE})}{I(x) \vdash \nu S(x)}\,(\text{SE})
$$

Induction hypothesis

Induction hypothesis

Induction hypothesis

Forward criterion [Sheeran+ 2000]

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Trying to make cycles after k-th iteration of (SE) rule

$$
\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \dfrac{\boxed{\overset{\text{SMT}}{\varphi_{k+1}(x) \vdash \bigvee_{i=1}^{k} \varphi_i(x)}} \qquad \dfrac{\varphi_k(x) \vdash \nu S(x) \quad \dots \quad \varphi_1(x) \overset{?}{\vdash} \nu S(x)}{\bigvee_{i=1}^{k} \varphi_i(x) \vdash \nu S(x)}}{\varphi_{k+1}(x) \vdash \nu S(x)}\,(\textsc{Cut})}{\varphi_k(x) \vdash \nu S(x)}\,(\mathrm{SE})
$$

Induction hypothesis

$$
\vdots
$$

$$
\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \vdots}{\varphi_2(x) \vdash \nu S(x)}\,(\mathrm{SE})}{\varphi_1(x) \vdash \nu S(x)}\,(\mathrm{SE})}{I(x) \vdash \nu S(x)}\,(\mathrm{SE})
$$

Induction hypothesis

Induction hypothesis

Forward criterion [Sheeran+ 2000]

Heuristic 1 Try to fit the shape of unproved sequents into the form $\varphi(x) \vdash \nu S(x)$

Trying to make cycles after k-th iteration of (SE) rule

$$
\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \vdots}{\varphi_2(x) \vdash \nu S(x)}\,(\mathrm{SE})}{\varphi_1(x) \vdash \nu S(x)}\,(\mathrm{SE})}{I(x) \vdash \nu S(x)}\,(\mathrm{SE})
$$

$$
\vdots
$$

$$
\dfrac{\overset{\text{SMT}}{\varphi_2(x) \vdash \neg B(x)} \qquad \dfrac{\boxed{\overset{\text{SMT}}{\varphi_{k+1}(x) \vdash \bigvee_{i=1}^{k} \varphi_i(x)}} \qquad \dfrac{\varphi_k(x) \vdash \nu S(x) \quad \cdots \quad \varphi_1(x) \vdash \nu S(x)}{\bigvee_{i=1}^{k} \varphi_i(x) \vdash \nu S(x)}}{\varphi_{k+1}(x) \vdash \nu S(x)}\,(\textsc{Cut})}{\varphi_k(x) \vdash \nu S(x)}\,(\mathrm{SE})
$$

Induction hypothesis

Induction hypothesis

Induction hypothesis

More aggressive use of (Cut)

$$\dfrac{\varphi(x) \vdash \neg B(x) \qquad \exists x.\varphi(x) \wedge T(x,y) \vdash \psi(y) \qquad \psi(y) \vdash \nu S(y)}{\varphi(x) \vdash \nu S(x)}\,(\mathrm{SE{+}C{\scriptsize UT}})$$

$$\left(\dfrac{\varphi(x) \vdash \neg B(x) \qquad \dfrac{\exists x.\varphi(x) \wedge T(x,y) \vdash \psi(y) \qquad \psi(y) \vdash \nu S(y)}{\exists x.\varphi(x) \wedge T(x,y) \vdash \nu S(y)}\,(\mathrm{C{\scriptsize UT}})}{\varphi(x) \vdash \nu S(x)}\,(\mathrm{SE})\right)$$

Question **How to select the cut formula ψ?**

Let Ξ be a finite set of formulas (closed under certain logical operations)

Heuristic 2 Let the cut formula be the **strongest** $\psi \in \Xi$ s.t. $\exists x.\, \varphi(x) \wedge T(x, y) \vdash \psi(y)$

⇒ **Predicate abstraction** [Ball+2001] [Graf&Saïdi 1997]

IMPACT [McMillan 2006]

Heuristic 3 Tentatively choose $\top$ **as the cut formula**

Heuristic 4 **When the proof attempt fails, strengthen the cut formulas** as follows

- **Replace cut formula φ_i with $\varphi_i \land Q_i$** and **solve the constraints on Q_i**

$$I(x) \overset{?}{\vdash} \nu S(x)$$

IMPACT [McMillan 2006]

Heuristic 3 Tentatively choose ⊤ **as the cut formula**

Heuristic 4 **When the proof attempt fails, strengthen the cut formulas** as follows

- **Replace cut formula φ_i with $\varphi_i \land Q_i$** and **solve the constraints on Q_i**

$$\dfrac{I(x) \overset{?}{\vdash} \neg B(x) \qquad \exists x. I(x) \land T(x,y) \overset{?}{\vdash} \boxed{\top} \qquad \boxed{\top} \overset{?}{\vdash} \nu S(y)}{I(x) \vdash \nu S(x)} \;(\text{SE+Cut})$$

IMPACT [McMillan 2006]

Heuristic 3 Tentatively choose ⊤ **as the cut formula**

Heuristic 4 **When the proof attempt fails, strengthen the cut formulas** as follows

- **Replace cut formula φ_i with $\varphi_i \land Q_i$** and **solve the constraints on Q_i**

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \exists x. I(x) \land T(x,y) \vdash^{?} \boxed{\top} \qquad \boxed{\top} \vdash^{?} \nu S(y)}{I(x) \vdash \nu S(x)}\ (\text{SE+Cut})$$

IMPACT [McMillan 2006]

Heuristic 3 Tentatively choose ⊤ **as the cut formula**

Heuristic 4 **When the proof attempt fails, strengthen the cut formulas** as follows

- **Replace cut formula φ_i with $\varphi_i \wedge Q_i$** and **solve the constraints on Q_i**

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \overset{\text{SMT}}{\exists x. I(x) \wedge T(x,y) \vdash \boxed{\top}} \qquad \boxed{\top} \overset{?}{\vdash} \nu S(y)}{I(x) \vdash \nu S(x)} \;(\text{SE+}\textsc{Cut})$$

IMPACT [McMillan 2006]

Heuristic 3 Tentatively choose $\top$ **as the cut formula**

Heuristic 4 **When the proof attempt fails, strengthen the cut formulas** as follows

- **Replace cut formula φ_i with $\varphi_i \wedge Q_i$** and **solve the constraints on Q_i**

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \overset{\text{SMT}}{\exists x. I(x) \wedge T(x,y) \vdash \top} \qquad \overset{?}{\top \vdash \nu S(y)}}{I(x) \vdash \nu S(x)} \ (\text{SE+Cut})$$

IMPACT [McMillan 2006]

Heuristic 3 Tentatively choose $\top$ **as the cut formula**

Heuristic 4 **When the proof attempt fails, strengthen the cut formulas** as follows

- **Replace cut formula φ_i with $\varphi_i \wedge Q_i$** and **solve the constraints on Q_i**

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \overset{\text{SMT}}{\exists x. I(x) \wedge T(x,y) \vdash \top} \qquad \dfrac{\overset{\textbf{False}}{\top \vdash \neg B(y)} \qquad \cdots\cdots}{\top \vdash \nu S(y)}\,(\mathrm{SE{+}C{\small UT}})}{I(x) \vdash \nu S(x)}\,(\mathrm{SE{+}C{\small UT}})$$

IMPACT [McMillan 2006]

Heuristic 3 Tentatively choose $\top$ **as the cut formula**

Heuristic 4 **When the proof attempt fails, strengthen the cut formulas** as follows

- **Replace cut formula φ_i with $\varphi_i \wedge Q_i$** and **solve the constraints on Q_i**

$$
\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \quad \exists x. I(x) \wedge T(x,y) \overset{?}{\vdash} \boxed{\top \wedge Q_1(y)} \quad \dfrac{\boxed{\top \wedge Q_1(y)} \overset{?}{\vdash} \neg B(y) \quad \cdots\cdots}{\boxed{\top \wedge Q_1(y)} \vdash \nu S(y)}\,(\text{SE+Cut})}{I(x) \vdash \nu S(x)}\,(\text{SE+Cut})
$$

IMPACT [McMillan 2006]

Heuristic 3 Tentatively choose ⊤ **as the cut formula**

Heuristic 4 **When the proof attempt fails, strengthen the cut formulas** as follows

- **Replace cut formula φ_i with $\varphi_i \wedge Q_i$** and **solve the constraints on Q_i**

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \exists x. I(x) \wedge T(x,y) \overset{?}{\vdash} \boxed{\top \wedge Q_1(y)} \qquad \dfrac{\boxed{\top \wedge Q_1(y)} \overset{?}{\vdash} \neg B(y) \quad \cdots\cdots}{\boxed{\top \wedge Q_1(y)} \vdash \nu S(y)}\,(\textsc{SE+Cut})}{I(x) \vdash \nu S(x)}\,(\textsc{SE+Cut})$$

$$\text{Constraints: } \{\exists x. I(x) \wedge T(x,y) \vdash \top \wedge Q_1(y), \quad \top \wedge Q_1(y) \vdash \neg B(y)\}$$

IMPACT [McMillan 2006]

Heuristic 3 Tentatively choose ⊤ **as the cut formula**

Heuristic 4 **When the proof attempt fails, strengthen the cut formulas** as follows

- **Replace cut formula φ_i with $\varphi_i \land Q_i$** and **solve the constraints on Q_i**

$$
\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \exists x. I(x) \land T(x,y) \overset{?}{\vdash} \boxed{\top \land Q_1(y)} \qquad \dfrac{\boxed{\top \land Q_1(y)} \overset{?}{\vdash} \neg B(y) \quad \cdots\cdots}{\boxed{\top \land Q_1(y)} \vdash \nu S(y)}\,(\text{SE+C}\textsc{ut})}{I(x) \vdash \nu S(x)}\,(\text{SE+C}\textsc{ut})
$$

$$
\text{Constraints: } \{\exists x. I(x) \land T(x,y) \vdash \top \land Q_1(y), \quad \top \land Q_1(y) \vdash \neg B(y)\}
$$

A solution of this constraint set is called an **interpolant**

Property-directed reachability

[Bradley 2011] [Een+ 2011] [Cimatti&Griggio 2012] ...

Heuristic 5 In strengthening, **keep cut formulas unchanged as many as possible**

$$\frac{\overset{\textbf{fail}}{\phi_n(x) \vdash \neg B(x)} \qquad \exists x.\varphi_n(x) \wedge T(x,y) \overset{?}{\vdash} \top \qquad \top \overset{?}{\vdash} \nu S(y)}{\varphi_n(y) \vdash \nu S(y)}\ (\text{SE+Cut})$$

$$\frac{\overset{\text{SMT}}{\varphi_{n-1}(x) \vdash \neg B(x)} \qquad \overset{\text{SMT}}{\exists x.\varphi_{n-1}(x) \wedge T(x,y) \vdash \varphi_n(y)} \qquad \varphi_n(y) \vdash \nu S(y)}{\varphi_{n-1}(y) \vdash \nu S(y)}\ (\text{SE+Cut})$$

$$\vdots$$

$$\frac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \overset{\text{SMT}}{\exists x.\varphi_1(x) \wedge T(x,y) \vdash \varphi_2(y)} \qquad \varphi_2(y) \vdash \nu S(y)}{\varphi_1(y) \vdash \nu S(y)}\ (\text{SE+Cut})$$

$$\frac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \overset{\text{SMT}}{\exists x.I(x) \wedge T(x,y) \vdash \varphi_1(y)} \qquad \varphi_1(y) \vdash \nu S(y)}{I(x) \vdash \nu S(x)}\ (\text{SE+Cut})$$

Property-directed reachability

[Bradley 2011] [Een+ 2011]
[Cimatti&Griggio 2012] ...

<u>Heuristic 5</u> In strengthening, **keep cut formulas unchanged as many as possible**

fail

?

?

$$\dfrac{\phi_n(x) \vdash \neg B(x) \qquad \exists x.\varphi_n(x) \wedge T(x,y) \vdash \top \qquad \top \vdash \nu S(y)}{\varphi_n(y) \vdash \nu S(y)}\ (\textsc{SE+Cut})$$

SMT

SMT

$$\dfrac{\varphi_{n-1}(x) \vdash \neg B(x) \qquad \exists x.\varphi_{n-1}(x) \wedge T(x,y) \vdash \boxed{\varphi_n(y)} \qquad \boxed{\varphi_n(y)} \vdash \nu S(y)}{\varphi_{n-1}(y) \vdash \nu S(y)}\ (\textsc{SE+Cut})$$

$$\vdots$$

SMT

SMT

$$\dfrac{\varphi_1(x) \vdash \neg B(x) \qquad \exists x.\varphi_1(x) \wedge T(x,y) \vdash \varphi_2(y) \qquad \varphi_2(y) \vdash \nu S(y)}{\varphi_1(y) \vdash \nu S(y)}\ (\textsc{SE+Cut})$$

SMT

SMT

$$\dfrac{I(x) \vdash \neg B(x) \qquad \exists x.I(x) \wedge T(x,y) \vdash \varphi_1(y) \qquad \varphi_1(y) \vdash \nu S(y)}{I(x) \vdash \nu S(x)}\ (\textsc{SE+Cut})$$

Property-directed reachability

[Bradley 2011] [Een+ 2011] [Cimatti&Griggio 2012] …

Heuristic 5 In strengthening, **keep cut formulas unchanged as many as possible**

fail

$$\phi_n(x) \vdash \neg B(x) \qquad \exists x.\varphi_n(x) \wedge T(x,y) \overset{?}{\vdash} \top \qquad \top \overset{?}{\vdash} \nu S(y)$$

(SE+Cut)

SMT

$$\varphi_{n-1}(x) \vdash \neg B(x) \qquad \exists x.\varphi_{n-1}(x) \wedge T(x,y) \vdash \boxed{\varphi_n(y)} \qquad \boxed{\varphi_n(y)} \vdash \nu S(y)$$

SMT

(SE+Cut)

$$\boxed{\varphi_{n-1}(y)} \vdash \nu S(y)$$

$$\vdots$$

SMT

$$\varphi_1(x) \vdash \neg B(x) \qquad \exists x.\varphi_1(x) \wedge T(x,y) \vdash \varphi_2(y) \qquad \varphi_2(y) \vdash \nu S(y)$$

SMT

(SE+Cut)

SMT

$$I(x) \vdash \neg B(x) \qquad \exists x.I(x) \wedge T(x,y) \vdash \varphi_1(y) \qquad \varphi_1(y) \vdash \nu S(y)$$

SMT

(SE+Cut)

$$I(x) \vdash \nu S(x)$$

Property-directed reachability

[Bradley 2011] [Een+ 2011] [Cimatti&Griggio 2012] …

Heuristic 5 In strengthening, **keep cut formulas unchanged as many as possible**

fail

$\phi_n(x) \vdash \neg B(x) \qquad \exists x.\varphi_n(x) \wedge T(x,y) \overset{?}{\vdash} \top \qquad \top \overset{?}{\vdash} \nu S(y)$

(SE+CUT)

$$\varphi_n(y) \vdash \nu S(y)$$

SMT SMT

$\varphi_{n-1}(x) \vdash \neg B(x) \qquad \exists x.\varphi_{n-1}(x) \wedge T(x,y) \vdash \boxed{\varphi_n(y)} \qquad \boxed{\varphi_n(y)} \vdash \nu S(y)$

(SE+CUT)

$$\boxed{\varphi_{n-1}(y)} \vdash \nu S(y)$$

$$\vdots$$

SMT SMT

$\varphi_1(x) \vdash \neg B(x) \qquad \exists x.\varphi_1(x) \wedge T(x,y) \vdash \boxed{\varphi_2(y)} \qquad \boxed{\varphi_2(y)} \vdash \nu S(y)$

(SE+CUT)

SMT SMT

$I(x) \vdash \neg B(x) \qquad \exists x.I(x) \wedge T(x,y) \vdash \varphi_1(y) \qquad \varphi_1(y) \vdash \nu S(y)$

(SE+CUT)

$$I(x) \vdash \nu S(x)$$

Property-directed reachability

[Bradley 2011] [Een+ 2011] [Cimatti&Griggio 2012] …

Heuristic 5 In strengthening, **keep cut formulas unchanged as many as possible**

$$\dfrac{\overset{\textbf{fail}}{\phi_n(x) \vdash \neg B(x)} \qquad \exists x.\varphi_n(x) \wedge T(x,y) \overset{?}{\vdash} \top \qquad \top \overset{?}{\vdash} \nu S(y)}{\boxed{\varphi_n(y)} \vdash \nu S(y)}\ (\mathrm{SE{+}Cut})$$

$$\dfrac{\overset{\text{SMT}}{\varphi_{n-1}(x) \vdash \neg B(x)} \qquad \exists x.\varphi_{n-1}(x) \wedge T(x,y) \overset{\text{SMT}}{\vdash} \boxed{\varphi_n(y)} \qquad \boxed{\varphi_n(y)} \vdash \nu S(y)}{\boxed{\varphi_{n-1}(y)} \vdash \nu S(y)}\ (\mathrm{SE{+}Cut})$$

$$\vdots$$

$$\dfrac{\overset{\text{SMT}}{\varphi_1(x) \vdash \neg B(x)} \qquad \exists x.\varphi_1(x) \wedge T(x,y) \overset{\text{SMT}}{\vdash} \boxed{\varphi_2(y)} \qquad \boxed{\varphi_2(y)} \vdash \nu S(y)}{\boxed{\varphi_1(y)} \vdash \nu S(y)}\ (\mathrm{SE{+}Cut})$$

$$\dfrac{\overset{\text{SMT}}{I(x) \vdash \neg B(x)} \qquad \exists x.I(x) \wedge T(x,y) \overset{\text{SMT}}{\vdash} \boxed{\varphi_1(y)} \qquad \boxed{\varphi_1(y)} \vdash \nu S(y)}{I(x) \vdash \nu S(x)}\ (\mathrm{SE{+}Cut})$$

Property-directed reachability

[Bradley 2011] [Een+ 2011] [Cimatti&Griggio 2012] ...

Heuristic 5 In strengthening, **keep cut formulas unchanged as many as possible**

In terms of constraints, **Heuristic 5** requires us to find a solution σ such that

$$\sigma(Q_1) = \cdots = \sigma(Q_k) = \top$$

for the largest possible k

$$\left\{ \begin{array}{ll} \exists x. I(x) \land T(x,y) \vdash \varphi_1(y) \land Q_1(y), & \varphi_1(y) \land Q_1(y) \vdash \neg B(y) \\ \exists x. \varphi_1(x) \land Q_1(x) \land T(x,y) \vdash \varphi_2(y) \land Q_2(y), & \varphi_2(y) \land Q_2(y) \vdash \neg B(y) \\ \qquad \vdots & \\ \exists x. \varphi_{n-1}(x) \land Q_{n-1}(x) \land T(x,y) \vdash \varphi_n(y) \land Q_n(y), & \varphi_n(y) \land Q_n(y) \vdash \neg B(y) \end{array} \right\}$$

Property-directed reachability

[Bradley 2011] [Een+ 2011] [Cimatti&Griggio 2012] …

Heuristic 5 In strengthening, **keep cut formulas unchanged as many as possible**

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Heuristic 5 In strengthening, **keep cut formulas unchanged as many as possible**

maximally conservative

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Heuristic 5 In strengthening, **keep cut formulas unchanged as many as possible**

maximally conservative

≈ Keep as many parts as possible unchanged

Similar ideas can be found in

- a procedure for game solving [Farzan&Kincaid 2017]
- **Spacer**, a state-of-the-art solver for non-linear CHCs [Komuravelli+, 2014]

Game solving [Farzan&Kincaid 2017]

Heuristic 5' In strengthening, **keep as many parts as possible unchanged**

They developed a validity checker for **first-order real arithmetic with ν**

- Corresponding to possibly infinite games with trivial condition

To prove $\vdash \nu X.\lambda x.\phi$, it constructs proofs of approximations $\vdash \nu^{(n)} X.\lambda x.\phi$ for $n = 1,2,\ldots$

- where $\nu^{(0)} X.\lambda x.\phi \coloneqq \top$ and $\nu^{(n+1)} X.\lambda x.\phi \coloneqq \phi[X \mapsto \nu^{(n)} X.\lambda x.\phi]$
- a proof of $\vdash \nu^{(n)} X.\lambda x.\phi$ can be seen as a partial proof of $\vdash \nu X.\lambda x.\phi$

The proof of $\vdash \nu^{(n+1)} X.\lambda x.\phi$ is adapted from the proof of Π of $\vdash \nu^{(n)} X.\lambda x.\phi$

- Replace every $\psi_0 \vdash \nu^{(0)} X.\lambda x.\phi$ in Π with a proof of $\psi_0 \vdash \nu^{(1)} X.\lambda x.\phi$
- If it fails, replace every $\psi_1 \vdash \nu^{(1)} X.\lambda x.\phi$ in Π with a proof of $\psi_1 \vdash \nu^{(2)} X.\lambda x.\phi$
- If it fails, ...

Spacer [Komuravelli+, 2014]

Heuristic 5' In strengthening, **keep as many parts as possible unchanged**

A solver for **non-linear CHCs**

$$\{I(x) \implies P(x), \quad P(x) \land P(y) \land T(x,y,z) \implies P(z), \quad P(z) \implies \neg B(z)\}$$

It has a solution iff $\left(\mu X. \lambda z. I(z) \lor \left(\exists xy. X(x) \land X(y) \land T(x,y,z)\right)\right)(z) \vDash \neg B(z)$

It constructs proofs of approximations $(\mu^{(n)} X. \ldots)(z) \vdash \neg B(z)$ for $n = 1,2, \ldots$

The proof of $(\mu^{(n+1)} X. \ldots)(z) \vdash \neg B(z)$ is adapted from $(\mu^{(n)} X. \ldots)(z) \vdash \neg B(z)$

- Construct the following proof, and try to strengthen the conclusion to $\ldots \vdash \neg B(z)$ by the "**smallest**" change

$$\dfrac{\begin{array}{c}\vdots\\ (\mu^{(n)} X. \ldots)(x) \vdash \neg B(x)\end{array} \qquad \begin{array}{c}\vdots\\ (\mu^{(n)} X. \ldots)(y) \vdash \neg B(y)\end{array} \qquad \neg B(x) \land \neg B(y) \land T(x,y,z) \vdash \top}{(\mu^{(n+1)} X. \ldots)(x) \vdash \top}$$

Spacer [Komuravelli+, 2014]

<u>Heuristic 5'</u> In strengthening, **keep as many parts as possible unchanged**

A solver for **non-linear CHCs**

$$\{I(x) \Longrightarrow P(x), \quad P(x) \wedge P(y) \wedge T(x,y,z) \Longrightarrow P(z), \quad P(z) \Longrightarrow \neg B(z)\}$$

It has a solution iff $\left(\mu X.\lambda z.I(z) \vee \left(\exists xy.X(x) \wedge X(y) \wedge T(x,y,z)\right)\right)(z) \vDash \neg B(z)$

It constructs proofs of approximations $(\mu^{(n)}X. \dots)(z) \vdash \neg B(z)$ for $n = 1,2, \dots$

The proof of $(\mu^{(n+1)}X. \dots)(z) \vdash \neg B(z)$ is adapted from $(\mu^{(n)}X. \dots)(z) \vdash \neg B(z)$

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Note on IC3/PDR and Spacer

Concrete methods for finding maximally conservative modifications are important

There are simple methods using **quantified formulas / quantifier elimination**

but **methods with quantifiers / QE are inefficient**

Both IC3/PDR and Spacer do not treat QE as a black box,
but **interleaving operations inside QE with those of the main procedure**

- During the computation of QE, one may obtain $\boldsymbol{QE(\exists x.\phi) = \psi_0} \lor ???$
- The detail of **the unknown part ??? may be irrelevant** to the main procedure
- So **we return to the main proc., freezing the computation of ???**
  - If it later turns out that ??? is important, we will resume that computation.

Future work

Other ideas in the verification community

- k-induction
- Splitter predicate and its generalizations
- Relational verification

Beyond the standard Hoare triples

- Total correctness, ω-regular, angelic nondeterminism, incorrectness
  - They have natural fixed-point encoding
- Verification of a procedural language (e.g., with angelic nondeterminism)
  - It does not seem to have natural encoding in first-order fixed-point logic

Dealing with ranking functions / disjunctively well-founded relations

Future work

Leveraging the characteristics of cyclic proofs

- Simpler invariants (cf. [Das 2020])
- Natural appearance of disjunction

Integrating proof search with interpolating theorem prover or other subprocedures

- Cf. Spacer (integration of search with QE)

Conclusion

Software model-checking algorithms can be seen as **cyclic proof search strategies**

- The connection is rather straightforward

  once the goal sequence is appropriately set

  "All initial states are safe"

  $$I(x) \vdash \nu S(x)$$

  where $\nu S(x) \overset{\nu}{\Leftrightarrow} \neg B(x) \land \left(\forall y. T(x, y) \Rightarrow \nu S(y)\right)$

- Several algorithms can be **reconstructed from simple proof-search heuristics**
- The usefulness of the connection is demonstrated by
  - revealing an unexpected connection: **PDR $\approx$ an efficient game solving algorithm**